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2003 J. Phys. A: Math. Gen. 36 4667

(http://iopscience.iop.org/0305-4470/36/16/315)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 4667–4675 PII: S0305-4470(03)57282-1

# The Bäcklund transformation equations for the ultradiscrete KP equation

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Received 6 December 2002
Published 8 April 2003
Online at stacks.iop.org/JPhysA/36/4667

**Abstract**
The Bäcklund transformation for the ultradiscrete KP equation is proposed. An algorithm to eliminate variables from the ultradiscrete linear equations is proposed. The consistency condition for the Bäcklund transformation equations is obtained via the algorithm.

PACS numbers: 02.30.Ik, 05.65.+b

## 1. Introduction

The Bäcklund transformation equations play an important role in the study of integrable equations [1]. They generate the Bäcklund transformation which allows one to obtain the exact $N$-soliton solution of the nonlinear equation, and also provide the Lax pair. The integrable equation itself is given as a consistency condition of the Bäcklund transformation equations in these schemes.

On the one hand, the cellular automaton models which have the exact $N$-soliton solution have been found in the last decade [9]. In many cases, these soliton cellular automaton models are obtained as a special limit, known as the ultradiscrete limit, of the discrete integrable equations. Typically, the limiting procedure simply replaces the summation by the Max operator and the product by the summation. Despite this simple correspondence, the procedure causes some ambiguities concerning the Max operator. For example, we can remove the variable $x$ from the equation $a + x = b + x$, however, we cannot remove the variable $x$ from

$$\mathrm{Max}(a, x) = \mathrm{Max}(b, x).$$

These types of difficulties prevent us from considering the consistency condition of the Bäcklund transformation equations for the soliton cellular automaton model.

In this paper, we consider the ultradiscrete KP equation and its Bäcklund transformation equations, since it reduces to many soliton cellular automaton models. The ultradiscrete KP equation was first proposed in [8], as the unified equation of a class of generalized box and ball systems. It also contains the ultradiscrete Toda equation [9], etc.

We show the consistency condition of the Bäcklund transformation equations of the ultradiscrete KP equation as follows. First, we propose an extension of the Bäcklund transformation equations of the discrete KP equation [2]. An important aspect of the extended Bäcklund transformation equations is that they can be expressed as the vanishing of the product of a matrix and a vector. The consistency condition becomes the vanishing of the determinant of the matrix. This will be shown in the next section. Second, in section 3, we show the algorithm to eliminate $N$ variables from $N$ ultradiscrete linear equations, before considering the ultradiscretization of the discrete KP equation. Finally, in section 4, we derive the Bäcklund transformation equations of the ultradiscrete KP equation, starting from the extended Bäcklund transformation equations of section 2. By applying the algorithm of section 3, we derive the consistency condition of the Bäcklund transformation equations. Some examples of soliton solutions are also shown.

## 2. Bäcklund transformation equation for the discrete KP equation

Let us begin by writing down the discrete KP equation for a function $f$ of three discrete variables $p, q, r$ [10, 11],

$$z_p z_{qr} f_p f_{qr} + z_q z_{rp} f_q f_{pr} + z_r z_{pq} f_r f_{pq} = 0. \tag{1}$$

Here, $z_i$ are arbitrary complex constants and $z_{ij} = z_i - z_j$. The subscripts of $f$ indicate the increase of the corresponding variables by one,

$$f_p = f(p+1, q, r) \qquad f_{pq} = f(p+1, q+1, r). \tag{2}$$

Despite the simplicity, this equation plays an important role in the study of the integrable equations. It reduces to many integrable equations known in physics and mathematics [10], and is satisfied by the transfer matrices of a class of solvable lattice models [5–7], and is also satisfied by the amplitude of a string model [4].

The Bäcklund transformation equations for the discrete KP equation are a system of linear equations

$$z_r f_r g_q - z_q f_q g_r + z_{qr} f_{qr} g = 0 \qquad z_r f_r g_p - z_p f_p g_r + z_{pr} f_{pr} g = 0. \tag{3}$$

If we substitute a solution of the discrete KP equation into the function $f$ in this equation, then we can show that there exist nontrivial solutions $g$, which satisfy the discrete KP equation. The compatibility condition of these two equations is at the centre of this scheme. It guarantees the existence of the solution for these equations, and it also guarantees that the solution $g$ also satisfies the discrete KP equation (see [3, 10] for example). However, the consistency condition does not work for the ultradiscretized system, because of the ambiguity of the Max operator stated above.

In this paper, we extend the Bäcklund transformation equations (3) by considering the additional bilinear equations deduced from the discrete KP equation and its Bäcklund transformation equations (3). The additional bilinear equations provide another expression of the Bäcklund transformation equations.

Let us rewrite the Bäcklund transformation equations by dividing them by $g$ and move the terms other than $f_{qr}$ and $f_{pr}$ to the right-hand side,

$$f_{qr} = \frac{-z_r f_r g_q + z_q f_q g_r}{z_{qr} g} \qquad f_{pr} = \frac{-z_r f_r g_p + z_p f_p g_r}{z_{pr} g}. \tag{4}$$

Substituting these expressions of $f_{qr}$ and $f_{pr}$ into the discrete KP equation, we can obtain an additional bilinear equation

$$z_p f_p g_q + z_q f_q g_p - z_{pq} f_{pq} g = 0. \tag{5}$$

Similarly, substituting $f_p$ and $f_q$ appearing in equations (3) into the discrete KP equation, we obtain another bilinear equation

$$z_{pq} f_{pq} g_r + z_{qr} f_{qr} g_p - z_{pr} f_{pr} g_q = 0. \tag{6}$$

Note that equations (3), (5) and (6) are linearly independent, unless the function $f$ satisfies the discrete KP equation. These four equations are the extension of the Bäcklund transformation equations, which we employ in this paper.

An important aspect of the extended Bäcklund transformation equations is that they can be summarized into the product of a matrix and a vector. Indeed, the extended Bäcklund transformation equations consist of four equations and the number of $g$ appearing in the same equations (i.e. $g, g_p, g_q, g_r$) is 4, thus, we can write down the equations by using a $4 \times 4$ matrix and a four-component vector

$$\begin{bmatrix} 0 & -z_r f_r & z_q f_q & z_{rq} f_{rq} \\ z_r f_r & 0 & -z_p f_p & z_{pr} f_{pr} \\ -z_q f_q & z_p f_p & 0 & z_{qp} f_{pq} \\ z_{qr} f_{qr} & z_{rp} f_{pr} & z_{pq} f_{pq} & 0 \end{bmatrix} \begin{bmatrix} g_p \\ g_q \\ g_r \\ g \end{bmatrix} = 0. \tag{7}$$

The consistency condition for these equations is the existence of the nontrivial solution, $g$, and it is nothing but the vanishing of the determinant of this matrix. In this case, the matrix is an antisymmetric matrix, thus, we can use the fact that the determinant of the antisymmetric matrix becomes the square of a Pfaffian. The vanishing of the Pfaffian just becomes the discrete KP equation.

Since the consistency condition is very different from the consistency condition for the original Bäcklund transformation equations, there is a chance to avoid the difficulty of the Max operator.

## 3. Ultradiscrete linear equations

Of course, we can remove the Max operator from the ultradiscrete Bäcklund transformation equations, by dividing it into cases. However, it is complicated and requires individual consideration. Instead, we study the generic consistency condition for the ultradiscrete linear equations. We show an algorithm to eliminate variables, starting from the consideration of two linear equations.

Before proceeding to the generic cases, let us consider the following two examples of two ultradiscrete linear equations. Our first example is the following couple of equations:

$$\tilde{a} + x = \mathrm{Max}(a + x, b + y) \qquad \tilde{c} + x = \mathrm{Max}(c + x, d + y). \tag{8}$$

Here, $a, b, c, d, \tilde{a}, \tilde{c}$ are the constant coefficients, and $x, y$ are the variables of these two equations. We can eliminate these variables by considering the trivial identity

$$\mathrm{Max}(\mathrm{Max}(a + x, b + y) + d, c + b + x) = \mathrm{Max}(a + d + x, \mathrm{Max}(c + x, d + y) + b). \tag{9}$$

Substituting the first and second equations of (8) into the left-hand side and the right-hand side of this equation respectively, we obtain the relation

$$\mathrm{Max}(\tilde{a} + d, c + b) = \mathrm{Max}(a + d, \tilde{c} + b). \tag{10}$$

This relation should be satisfied if there are finite solutions, $x$ and $y$. Thus, we can eliminate all variables, without dividing Max operators into the cases.

Our second example is the following couple of linear equations:

$$\tilde{a} + x = \mathrm{Max}(a + x, b + y) \qquad \tilde{d} + y = \mathrm{Max}(c + x, d + y). \tag{11}$$

We can also eliminate all variables from these equations without dividing them into cases. Indeed, adding these two equations and applying these two equations to the left-hand side once again, we obtain the relations

$$\begin{aligned} \tilde{a} + \tilde{d} + x + y &= \mathrm{Max}(a + c + 2x, a + d + x + y, b + c + x + y, b + d + 2y) \\ &= \mathrm{Max}(a + \tilde{d} + x + y, \tilde{a} + d + x + y, b + c + x + y). \end{aligned} \tag{12}$$

We can eliminate all variables by subtracting the $x + y$ from the first part and the third part of these equations,

$$\mathrm{Max}(\tilde{a} + d, a + \tilde{d}, b + c) = \tilde{a} + \tilde{d}. \tag{13}$$

In the general two linear equations cases, we cannot eliminate all variables without dividing them into cases. However, the previous two examples make the consideration simple. The most general ultradiscrete two linear equations for two variables $x$ and $y$ are written as

$$\mathrm{Max}(\tilde{a} + x, \tilde{b} + y) = \mathrm{Max}(a + x, b + y) \qquad \mathrm{Max}(\tilde{c} + x, \tilde{d} + y) = \mathrm{Max}(c + x, d + y). \tag{14}$$

For example, if $\tilde{a} + x$ on the left-hand side of the first equation is larger than $\tilde{b} + y$, and $\tilde{c} + x$ in the second equation is larger than $\tilde{d} + y$, these equations become

$$\tilde{a} + x = \mathrm{Max}(a + x, b + y) \qquad \tilde{c} + x = \mathrm{Max}(c + x, d + y). \tag{15}$$

This is the case of our first example. By applying the result of the first example, we can eliminate all variables from these equations.

$$\mathrm{Max}(\tilde{a} + d, c + b) = \mathrm{Max}(a + d, \tilde{c} + b). \tag{16}$$

Applying a similar argument to the rest of the three cases

$$\begin{aligned} &\tilde{a} + x \geqslant \tilde{b} + y \text{ and } \tilde{c} + x \geqslant \tilde{d} + y \quad \Rightarrow \quad \mathrm{Max}(\tilde{a} + d, c + b) = \mathrm{Max}(a + d, \tilde{c} + b) \\ &\tilde{a} + x \leqslant \tilde{b} + y \text{ and } \tilde{c} + x \leqslant \tilde{d} + y \quad \Rightarrow \quad \mathrm{Max}(\tilde{b} + c, d + a) = \mathrm{Max}(b + c, \tilde{d} + a) \\ &\tilde{a} + x \geqslant \tilde{b} + y \text{ and } \tilde{c} + x \leqslant \tilde{d} + y \quad \Rightarrow \quad \tilde{a} + \tilde{d} = \mathrm{Max}(\tilde{a} + d, a + \tilde{d}, b + c) \\ &\tilde{a} + x \leqslant \tilde{b} + y \text{ and } \tilde{c} + x \geqslant \tilde{d} + y \quad \Rightarrow \quad \tilde{b} + \tilde{c} = \mathrm{Max}(\tilde{b} + c, b + \tilde{c}, a + d) \end{aligned} \tag{17}$$

and summarizing these results in one equation, we can obtain the relation

$$\mathrm{Max}(a + d, \tilde{a} + \tilde{d}, b + \tilde{c}, \tilde{b} + c) = \mathrm{Max}(\tilde{a} + d, a + \tilde{d}, b + c, \tilde{b} + \tilde{c}). \tag{18}$$

One can easily confirm that this relation is satisfied in all four cases. This relation should be satisfied if there are finite solutions, $x$ and $y$, for the generic two linear equations (14).

On the one hand, we can rewrite equations (14) as

$$\mathrm{Max}(\tilde{a} + z, \tilde{b}) = \mathrm{Max}(a + z, b) \qquad \mathrm{Max}(\tilde{c} + z, \tilde{d}) = \mathrm{Max}(c + z, d). \tag{19}$$

Here, we subtract $y$ from equations (14), and replace $x - y$ by $z$. Apparently, if there is a finite $z$ which satisfies these equations, relation (18) should be satisfied. Since $a, b, c, d, \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ are arbitrary constants, we can eliminate one variable from two linear equations of any number of variables. Applying this reduction recursively, we can reduce $N$ linear equations for $N$ variables to two linear equations for two variables, and the remaining two variables can be eliminated from two linear equations, so that we can eliminate $N$ variables from $N$ linear equations.

In the next section, we consider the consistency condition of the Bäcklund transformation equations for the ultradiscrete KP equation by using this algorithm.

## 4. The Bäcklund transformation equations for the ultradiscrete KP equation

Let us write the discrete KP equation once again. It was as follows:

$$z_p z_{qr} f_p f_{qr} + z_r z_{pq} f_r f_{pq} = z_q z_{pr} f_q f_{pr}. \tag{20}$$

Here, $f$ is a complex valued function in the original discrete KP equation, but we restrict the value to positive real numbers, to take the ultradiscrete limit. We also restrict the values of $z_i$ to real numbers, and assume $z_p > z_q > z_r$ since it does not lose generality. Note that in this choice of coefficients, both the left-hand side and the right-hand side of equation (20) become positive.

The ultradiscrete limit of the discrete KP equation is taken as follows. First, we transform the dependent variable $f$ and the coefficients $z_i$, $z_{ij}$ into $F$ and $Z_i$, $Z_{ij}$, respectively, as

$$f = \exp\left(\frac{F}{\epsilon}\right) \qquad z_i = \exp\left(\frac{Z_i}{\epsilon}\right) \qquad z_{ij} = \exp\left(\frac{Z_{ij}}{\epsilon}\right). \tag{21}$$

Here, $\epsilon$ is a positive real parameter. Second, we apply the log function to both sides of equation (20) after multiplication of $\epsilon$, and take the $\epsilon \to +0$ limit. By using the following two identities,

$$\begin{aligned} &\lim_{\epsilon\to+0} \log\epsilon\left(\exp\left(\frac{A}{\epsilon}\right) + \exp\left(\frac{B}{\epsilon}\right)\right) = \mathrm{Max}(A, B) \\ &\lim_{\epsilon\to+0} \log\epsilon\left(\exp\left(\frac{A}{\epsilon}\right) \times \exp\left(\frac{B}{\epsilon}\right)\right) = A + B \end{aligned} \tag{22}$$

we can obtain the ultradiscrete KP equation from equation (20):

$$Z_q + Z_{pr} + F_q + F_{pr} = \mathrm{Max}(Z_p + Z_{qr} + F_p + F_{qr}, Z_r + Z_{pq} + F_r + F_{pq}) \tag{23}$$

As in the case of the discrete KP equation, this equation reduces to many soliton cellular automaton models (see [8], for example).

The Bäcklund transformation equations for the ultradiscrete KP equation are obtained by applying a similar procedure to the extended Bäcklund transformation equations. Applying the transformation (21) and

$$g = \exp\left(\frac{G}{\epsilon}\right) \tag{24}$$

and taking the $\epsilon \to +0$ limit, we obtain the equations

$$\begin{aligned} Z_q + F_q + G_r &= \mathrm{Max}[Z_r + F_r + G_q, Z_{qr} + F_{qr} + G] \\ Z_p + F_p + G_r &= \mathrm{Max}[Z_r + F_r + G_p, Z_{pr} + F_{pr} + G] \\ Z_p + F_p + G_q &= \mathrm{Max}[Z_q + F_q + G_p, Z_{pq} + F_{pq} + G] \\ Z_{pr} + F_{pr} + G_q &= \mathrm{Max}[Z_{qr} + F_{qr} + G_p, Z_{pq} + F_{pq} + G_r]. \end{aligned} \tag{25}$$

The consistency condition for these equations is obtained by eliminating all the variables from these equations. Let us note that we can eliminate $G_r$ and $G_q$ by substituting the second equation and third equation into the first equation and fourth equation, respectively. Then, we obtain the following two linear equations for two variables $G_p$ and $G$:

$$\begin{aligned} &\mathrm{Max}(Z_r + Z_q + F_r + F_q + G_p, Z_q + Z_{pr} + F_q + F_{pr} + G) \\ &\quad = \mathrm{Max}(Z_r + Z_q + F_r + F_q + G_p, \mathrm{Max}(Z_r + Z_{pq} \\ &\qquad + F_r + F_{pq}, Z_{qr} + Z_p + F_{qr} + F_p) + G)) \\ &\mathrm{Max}(Z_{pr} + Z_q + F_{pr} + F_q + G_p, Z_{pr} + Z_{pq} + F_{pr} + F_{pq} + G) \\ &\quad = \mathrm{Max}(\mathrm{Max}(Z_{qr} + Z_p + F_{qr} + F_p, Z_{pq} + Z_r + F_{pq} + F_r) \\ &\qquad + G_p, Z_{pq} + Z_{pr} + F_{pq} + F_{pr} + G). \end{aligned} \tag{26}$$

By applying relation (18), we can eliminate the remaining two variables. After some calculation, we can obtain the equation for $F$:

$$\begin{aligned}\mathrm{Max}(Z_q &+ Z_r + Z_{pq} + Z_{pr} + F_q + F_r + F_{pq} + F_{pr}, Z_{pr} + Z_q + F_{pr} + F_q \\ &+ \mathrm{Max}(Z_r + Z_{pq} + F_r + F_{pq}, Z_{qr} + Z_p + F_{qr} + F_p), Z_q + Z_{pr} \\ &+ F_q + F_{pr} + \mathrm{Max}(Z_{qr} + Z_p + F_{qr} + F_p, Z_{pq} + Z_r + F_{pq} + F_r)) \\ &= \mathrm{Max}(Z_q + Z_r + Z_{pq} + Z_{pr} + F_q + F_r + F_{pq} + F_{pr}, 2(Z_{pr} + Z_q \\ &+ F_{pr} + F_q), 2\mathrm{Max}(Z_r + Z_{pq} + F_r + F_{pq}, Z_{qr} + Z_p + F_{qr} + F_p)).\end{aligned} \tag{27}$$

Note that we can remove the first terms on both sides, since they are smaller than the other terms. Moving the rest of the right-hand side to the left-hand side, we obtain the equation

$$\| Z_q + Z_{pr} + F_q + F_{pr} - \mathrm{Max}(Z_p + Z_{qr} + F_p + F_{qr}, Z_r + Z_{pq} + F_r + F_{pq}) \| = 0. \tag{28}$$

This is nothing but the absolute value of the ultradiscrete KP equation.

Since we can change the roles of $F$ and $G$, by shifting each of the equations of (25) appropriately, $G$ should also satisfy the ultradiscrete KP equation. Thus, we can show that equations (25) really generate the Bäcklund transformation of the ultradiscrete KP equation.

In the rest of this section, we derive the two-soliton solution and conjecture the form of the $N$-soliton solution. Let us assume the existence of the trivial solution, $F = G = 0$, then the conditions for the coefficients of equations (25)

$$Z_q = \mathrm{Max}[Z_r, Z_{qr}] \qquad Z_p = \mathrm{Max}[Z_r, Z_{qr}, Z_{pq}] \qquad Z_{pr} = \mathrm{Max}[Z_{qr}, Z_{pq}] \tag{29}$$

should be satisfied. Note that $Z_r$, $Z_{qr}$ and $Z_{pq}$ determine the rest of the coefficients. We rewrite these three coefficients as

$$Z_{pq} = Z_1 \qquad Z_{qr} = Z_2 \qquad Z_r = Z_3 \tag{30}$$

for simplicity. Furthermore, we assume the relations $Z_1 \leqslant Z_2 \leqslant Z_3$, then the rest of the coefficients become

$$Z_q = Z_2 \qquad Z_p = Z_1 \qquad Z_{pr} = Z_1. \tag{31}$$

Of course, we can consider the other relations for $Z_1$, $Z_2$ and $Z_3$, however, it does not change the following argument.

The one-soliton solution is derived as follows. First, we substitute the trivial solution $F = 0$ into the Bäcklund transformation equations (25), and assume the form of $G$ as

$$G = \vec{\alpha} \cdot \vec{p}. \tag{32}$$

Here, $\vec{\alpha}$ is the three-dimensional vector of constant coefficients and $\vec{p}$ is the vector of coordinates $p, q, r$,

$$\vec{\alpha} = (\alpha_p, \alpha_q, \alpha_r) \qquad \vec{p} = (p, q, r). \tag{33}$$

The dot between them represents the vector product. By substituting the $G$ of equation (32) into the Bäcklund transformation equations (25), we obtain the following dispersion relations:

$$\begin{aligned} Z_2 + \alpha_r &= \mathrm{Max}[Z_3 + \alpha_q, Z_2] \\ Z_1 + \alpha_r &= \mathrm{Max}[Z_3 + \alpha_p, Z_1] \\ Z_1 + \alpha_q &= \mathrm{Max}[Z_2 + \alpha_p, Z_1] \\ Z_1 + \alpha_q &= \mathrm{Max}[Z_2 + \alpha_p, Z_1 + \alpha_r]. \end{aligned} \tag{34}$$

We can solve these dispersion relations as

$$\alpha_p = \alpha + Z_1 \qquad \alpha_q = \mathrm{Max}(\alpha + Z_2, 0) \qquad \alpha_r = \mathrm{Max}(\alpha + Z_3, 0). \tag{35}$$

Here, $\alpha$ is an arbitrary real parameter. The one-soliton solution of the ultradiscrete KP equation becomes the superposition of these representations of $G$ which is written in the 'Max' operator,

$$G = \mathrm{Max}(\vec{\alpha} \cdot \vec{p},\ \vec{\beta} \cdot \vec{p}) \tag{36}$$

where $\beta$ is a three-dimensional vector $(\beta_p, \beta_q, \beta_r)$ which satisfies the dispersion relations.

Figure 1 shows the time evolution of three-soliton solution. We transform the dependent variable $f$ and coordinates $p, q, r$ into $U$ and $l, m, n$ as

$$\begin{aligned} &U(p,q,r) = f(p,q,r+1) + f(p+1,q+1,r) - f(p+1,q,r) - f(p,q+1,r+1) \\ &l = p - r \qquad m = q \qquad n = p + r. \end{aligned} \tag{37}$$

The $l$ axis is parallel to the horizontal line and the $m$ axis is parallel to the vertical line.

We see the propagation of one-soliton solution in the large $l$ and large $m$ region in figure 1.

Similarly, we can obtain the two-soliton solution by substituting the one-soliton solution (36) to the function $F$ in the Bäcklund transformation equations, and solve the equations for $G$. Let us assume the form of $G$ as

$$G = \mathrm{Max}(c_\alpha + \vec{\alpha} \cdot \vec{p} + \vec{\gamma} \cdot \vec{p}, c_\beta + \vec{\beta} \cdot \vec{p} + \vec{\gamma} \cdot \vec{p}) \tag{38}$$

where $c_\alpha$ and $c_\beta$ are the constants which should be determined later, and $\vec{\gamma}$ is also a three-dimensional vector which satisfies the dispersion relations. Substituting this expression into the Bäcklund transformation equations, we obtain the following relations of $c_\alpha$ and $c_\beta$:

$$\begin{cases} c_\alpha = c_\beta & \text{if} \quad \gamma \leqslant \alpha, \beta \\ c_\alpha - \alpha = c_\beta - \beta & \text{if} \quad \alpha, \beta \leqslant \gamma \\ \text{There is no solution of } c_\alpha \text{ and } c_\beta & \text{if} \quad \alpha \leqslant \gamma \leqslant \beta \quad \text{or} \quad \beta \leqslant \gamma \leqslant \alpha. \end{cases} \tag{39}$$

These results can be summarized in the following forms, without the loss of generality

$$\begin{aligned} &c_\alpha = \mathrm{Max}(\alpha, \gamma) \\ &c_\beta = \mathrm{Max}(\alpha, \gamma) \qquad (\alpha, \beta \leqslant \gamma \text{ or } \gamma \leqslant \alpha, \beta). \end{aligned} \tag{40}$$

The two-soliton solution of the ultradiscrete KP equation becomes the superposition of these solutions,

$$\begin{aligned} G = \mathrm{Max}(&\mathrm{Max}(\alpha, \gamma) + \vec{\alpha} \cdot \vec{p} + \vec{\gamma} \cdot \vec{p} \qquad \mathrm{Max}(\alpha, \delta) + \vec{\alpha} \cdot \vec{p} + \vec{\delta} \cdot \vec{p} \\ &\mathrm{Max}(\beta, \gamma) + \vec{\beta} \cdot \vec{p} + \vec{\gamma} \cdot \vec{p}, \mathrm{Max}(\beta, \gamma) + \vec{\beta} \cdot \vec{p} + \vec{\delta} \cdot \vec{p}) \end{aligned} \tag{41}$$

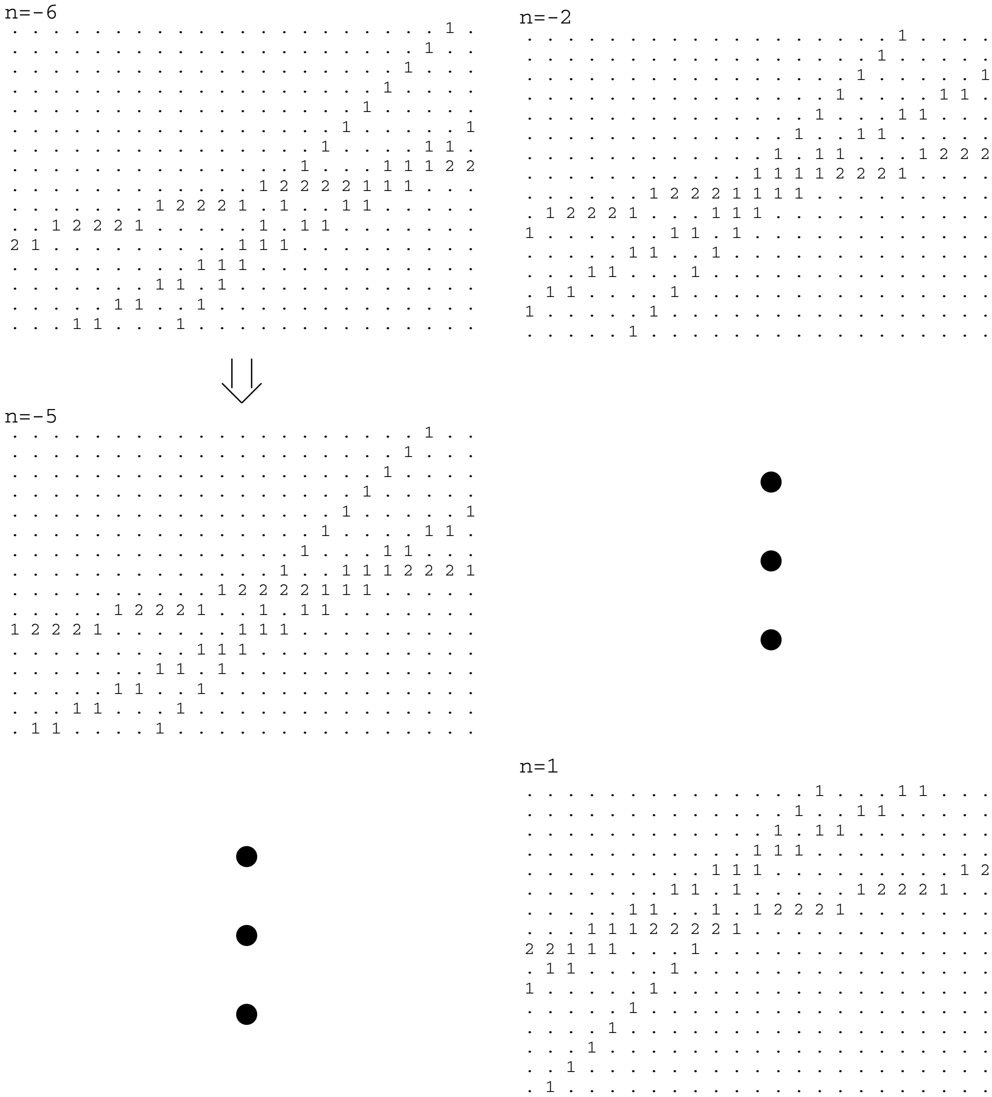
where $\gamma$ and $\delta$ should satisfy one of the following conditions:

$$\begin{aligned} &\gamma, \delta \leqslant \alpha, \beta \\ &\gamma \leqslant \alpha, \beta \leqslant \delta \\ &\alpha, \beta \leqslant \gamma, \delta. \end{aligned} \tag{42}$$

These arguments allow one to suppose the form of the $N$-soliton solution. The $N$-soliton solution may have the following form:

$$f = \mathrm{Max}_{\{\sigma\}} \left( \sum_{i<j} \mathrm{Max}\left(\alpha_i^{\sigma_i}, \alpha_j^{\sigma_j}\right) + \sum_i \vec{\alpha}_i^{\,\sigma_i} \cdot \vec{p} \right). \tag{43}$$

Here, $\{\sigma\} = (\sigma_1, \sigma_2, \ldots, \sigma_N)$ and $\sigma_i$ take the value of 1 or 0. $\mathrm{Max}_{\{\sigma\}}$ means to take the maximal value of all the combinations of the $\sigma$. The vectors $\vec{\alpha}_i^{\,0}$ and $\vec{\alpha}_i^{\,1}$ are constant coefficients which




$$Z_1 = 0\ ,\ Z_2 = 1\ ,\ Z_3 = 0,$$
$$\alpha_1^0 = 0\ ,\ \alpha_1^1 = -1,$$
$$\alpha_2^0 = 1\ ,\ \alpha_2^1 = -1,$$
$$\alpha_3^0 = 1\ ,\ \alpha_3^1 = -4.$$

**Figure 1.** Time evolution of the three-soliton solution.

satisfy the dispersion relations, and we also assume that one of the following conditions is satisfied for all combinations of $i \leqslant j$:

$$\alpha_i^0, \alpha_i^1 \leqslant \alpha_j^0, \alpha_j^1 \qquad \alpha_i^0 \leqslant \alpha_j^0, \alpha_j^1 \leqslant \alpha_j^1 \qquad \alpha_j^0, \alpha_j^1 \leqslant \alpha_i^0, \alpha_i^1. \tag{44}$$

We do not have the proof of expression (43), but we confirm that it really satisfies the Bäcklund transformation equations upto $N = 4$. The complete proof will be given in another paper.

## 5. Conclusion

We consider the consistency condition for the Bäcklund transformation equations of the ultradiscrete KP equation. We extend the Bäcklund transformation equations for the discrete KP equation into the form of the product of a matrix and a vector. The consistency condition becomes the vanishing of the determinant of the matrix. In this case, the matrix becomes an antisymmetric matrix, and we can obtain the discrete KP equation by using the relation between the determinant of an antisymmetric matrix and a Pfaffian. We also show the algorithm to eliminate the variables from the ultradiscrete linear equations. By using this algorithm, we obtain the consistency condition of the Bäcklund transformation equations of the ultradiscrete KP equation. The consistency condition becomes the absolute value of the ultradiscrete KP equation, instead of the square of the Pfaffian in the discrete KP equation case.

## References


[1] Rogers C and Shadowick W E 1982 *Bäcklund Transformations and their Applications* (New York: Academic)
[2] Shinzawa N and Saito S 1998 *J. Phys. A: Math. Gen.* **31** 4533–40
Shinzawa N 2000 *J. Phys. A: Math. Gen.* **33** 3957–70
[3] Saitoh N and Saito S 1987 *J. Phys. Soc. Japan* **56** 1664–74
[4] Saito S 1987 *Phys. Rev.* D **36** 1819–26
[5] MaCoy B, Perk J and Wu T T 1981 *Phys. Rev. Lett* **46** 757
[6] Kuniba A, Nakanishi T and Suzuki J 1994 *Int. J. Mod. Phys.* A **9** 5215
Kuniba A, Naknishi T and Suzuki J 1994 *Int. J. Mod. Phys.* A **9** 5267
[7] Krichever I, Lipan O, Wiegmann P and Zabrodin A 1996 *Commun. Math. Phys.* **188** 267
[8] Tokihiro T, Takahashi D and Matsukidaira J 2000 *J. Phys. A: Math. Gen.* **33** 607–19
[9] Takahashi D and Satsuma J 1990 *J. Phys. Soc. Japan* **59** 3514
Takahashi D and Matsukidaira J 1995 *Phys. Lett.* A **209** 184
Tokihiro T, Takahashi D, Matsukidaira J and Satsuma J *Phys. Rev. Lett.* **76** 3247
Matsukidaira J, Satsuma J, Takahashi D, Tokihiro T and Torii M 1997 *Phys. Lett.* A **225** 287–95
[10] Hirota R 1981 *J. Phys. Soc. Japan* **50** 3785–91
[11] Kashiwara M and Miwa T 1981 *Proc. Japan Acad.* **57** Ser. A
Date E, Jimbo M, Kashiwara M and Miwa T 1981 *J. Phys. Soc. Japan* **50** 3806–4812
Date E, Jimbo M and Miwa T 1982 *J. Phys. Soc. Japan* **51** 4125–31